Applied Mathematics Letters 23 (2010) 1346–1350

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Calculation of coefficients of a cardinal B-spline[☆]

Gradimir V. Milovanović[a,*], Zlatko Udovičić[b]

[a] Faculty of Computer Sciences, Megatrend University, Bulevar umetnosti 29, 11070 Novi Beograd, Serbia
[b] Faculty of Sciences, Department of Mathematics, Zmaja od Bosne 35, 71000 Sarajevo, Bosnia and Herzegovina

ARTICLE INFO

Article history:
Received 5 April 2010
Received in revised form 16 June 2010
Accepted 23 June 2010

Keywords:
Cardinal B-spline
Polynomial
Coefficients

ABSTRACT

It is well known that a cardinal B-spline of order m, $m \in \mathbb{N}$, is a piecewise polynomial function. In this note we propose an effective method for calculating the coefficients of polynomials which constitute a cardinal B-spline.

1. Introduction

Cardinal B-splines play a very important role in the approximation theory (different methods for solving initial and boundary value problems, spline interpolation, multi-resolution approximation, etc.). Roughly speaking, a cardinal B-spline of order m, $m \in \mathbb{N}$, is a real function with the following properties:

- its support is the interval $[0, m]$;
- it belongs to the class $C^{m-2}[0, m]$, $m \geq 2$;
- at each interval $[k, k + 1]$, $0 \leq k \leq m - 1$, it is a polynomial of degree $m - 1$.

In this short note we give an effective, simple and useful algorithm for calculating the coefficients of the mentioned polynomials.

The paper is organized as follows. In this section we give the definition of the cardinal B-spline and a list of its basic properties. The main result is given in the second part where we deduce recurrence relations for calculating the coefficients. Also, we describe an algorithm for realization of the obtained relations. Finally, in the last section we give numerical results, which explicitly describe cardinal B-splines of orders $2, 3, \ldots, 7$.

Definition 1.1. A cardinal B-spline of first order, denoted by $\varphi_1(\cdot)$, is a characteristic function of the interval $[0, 1)$, i.e.,

$$\varphi_1(x) = \begin{cases} 1, & x \in [0, 1), \\ 0, & \text{otherwise}. \end{cases}$$

A cardinal B-spline of order m, $m \in \mathbb{N}$, denoted by $\varphi_m(\cdot)$, is defined as a convolution

$$\varphi_m(x) = (\varphi_{m-1} * \varphi_1)(x) = \int_{\mathbb{R}} \varphi_{m-1}(x - t)\varphi_1(t)\,\mathrm{d}t = \int_0^1 \varphi_{m-1}(x - t)\,\mathrm{d}t.$$

☆ The first author was supported in part by the Serbian Ministry of Science and Technological Development (Project: Orthogonal Systems and Applications, grant number #144004).

* Corresponding author.
E-mail addresses: gvm@megatrend.edu.rs (G.V. Milovanović), zzlatko@pmf.unsa.ba (Z. Udovičić).

0893-9659/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2010.06.029

Theorem 1.2. *A cardinal B-spline of order m, $m \in \mathbb{N}$, has the following properties:*

1° supp $\varphi_m(\cdot) = [0, m]$;
2° $\varphi_m(\cdot) \in C^{m-2}[0, m]$;
3° *At each interval* $[k, k+1]$, $0 \le k \le m-1$, *the cardinal B-spline of order m is a polynomial of degree $m-1$;*
4° *For each $t \in [0, m]$*

$$\varphi_m(t) = \frac{t}{m-1}\varphi_{m-1}(t) + \frac{m-t}{m-1}\varphi_{m-1}(t-1), \quad m \ge 2, \tag{1.1}$$

$$\varphi_m'(t) = \varphi_{m-1}(t) - \varphi_{m-1}(t-1), \quad m \ge 2; \tag{1.2}$$

5° *A cardinal B-spline is symmetric on the interval* $[0, m]$, *i.e., for each* $t \in [0, m]$,

$$\varphi_m(t) = \varphi_m(m-t).$$

The proof of this theorem, as well as much more details on cardinal B-splines, can be found in [1] or [2].

2. Main result

Equalities (1.1) and (1.2), after some simplification, give the following differential equation

$$(m-x)\varphi_m'(x) + (m-1)\varphi_m(x) = m\varphi_{m-1}(x).$$

We will look for a solution of this equation (obviously it is a cardinal B-spline of order m) in a polynomial form, assuming that the coefficients of the cardinal B-spline of order $m-1$ are known. Hence, let $x \in [k, k+1]$, $k \in \mathbb{N}$, $1 \le k \le m-1$ and let $\varphi_m(x) = \sum_{i=0}^{m-1} a_i^{(m,k)} x^i$. Then $\varphi_m'(x) = \sum_{i=0}^{m-2} (i+1) a_{i+1}^{(m,k)} x^i$, and so

$$\begin{aligned}(m-x)\varphi_m'(x) + (m-1)\varphi_m(x) &= (m-x)\sum_{i=0}^{m-2}(i+1)a_{i+1}^{(m,k)}x^i + (m-1)\sum_{i=0}^{m-1}a_i^{(m,k)}x^i \\ &= \sum_{i=0}^{m-2}\left[m(i+1)a_{i+1}^{m,k} + (m-i-1)a_i^{(m,k)}\right],\end{aligned}$$

while on the other hand we have

$$m\varphi_{m-1}(x) = m\sum_{i=0}^{m-2} a_i^{(m-1,k)} x^i.$$

After identifying the corresponding coefficients, we obtain the following relations

$$\frac{m(i+1)}{i+1-m} a_{i+1}^{(m,k)} - a_i^{(m,k)} = \frac{m}{i+1-m} a_i^{(m-1,k)}, \quad 0 \le i \le m-2. \tag{2.3}$$

These relations are a system of $m-2$ linear equations with $m-1$ unknown coefficients. This system can be extended by the following equation

$$(m-1)\cdot k \cdot a_{m-1}^{(m,k)} + a_{m-2}^{(m,k)} = (m-1)\cdot k \cdot a_{m-1}^{(m,k-1)} + a_{m-2}^{(m,k-1)},$$

which is a consequence of the continuity of the derivative of order $m-2$ of the cardinal B-spline at the point $x = k$. The last two equations are a subsystem of second order and one solution of this subsystem is given by

$$a_{m-1}^{(m,k)} = \frac{1}{(m-1)(m-k)}\left[m a_{m-2}^{(m-1,k)} - k(m-1)a_{m-1}^{(m,k-1)} - a_{m-2}^{(m,k-1)}\right].$$

The remaining coefficients are obtained from (2.3) in the following way

$$a_i^{(m,k)} = \frac{m}{i+1-m}\left[(i+1)a_{i+1}^{(m,k)} - a_i^{(m-1,k)}\right], \quad m-2 \ge i \ge 0.$$

If $k = 0$, it is not possible to apply the described procedure, so in this case we use special relations. Hence, let $x \in [0, 1]$. From the fact that 0 is a root of the polynomial $\sum_{i=0}^{m-1} a_i^{(m,0)} x^i$ with multiplicity $m-2$, it immediately follows that

$$a_i^{(m,0)} = 0, \quad m-2 \ge i \ge 0.$$

To determine the coefficient $a_{m-1}^{(m,0)}$ we use the relation (2.3) (which also holds in the case $k = 0$) and obtain

$$a_{m-1}^{(m,0)} = \frac{1}{m-1} a_{m-2}^{(m-1,0)}.$$

The obtained relations should be used to calculate the coefficients of the cardinal B-spline at the "left hand side" of the interval $[0, m]$. To reduce the calculating process, for calculating the coefficients on the "right hand side" of the interval $[0, m]$ we use a symmetry of the cardinal B-spline. More precisely, we use a symmetry of its derivatives of the corresponding order. From the fact that $\varphi_m(x) = \varphi_m(m-x)$, it follows that

$$\varphi_m^{(j)}(x) = (-1)^j \varphi_m^{(j)}(m-x),$$

for all $x \in [0, m]$ and all $j \in \{0, 1, \ldots, m-1\}$.

Now, let $x \in [m-k-1, m-k]$. Then

$$\begin{aligned}
\varphi_m^{(j)}(x) &= \sum_{i=0}^{m-j-1} (i+1)(i+2)\cdots(i+j) a_{i+j}^{(m,m-k-1)} x^i \\
&= (-1)^j \varphi_m^{(j)}(m-x) \\
&= (-1)^j \sum_{i=0}^{m-j-1} (i+1)(i+2)\cdots(i+j) a_{i+j}^{(m,k)} (m-x)^i \\
&= (-1)^j \sum_{i=0}^{m-j-1} (i+1)(i+2)\cdots(i+j) a_{i+j}^{(m,k)} \sum_{\ell=0}^{i} \binom{i}{\ell} m^{i-\ell} (-1)^\ell x^\ell \\
&= (-1)^j \sum_{i=0}^{m-j-1} (i+1)(i+2)\cdots(i+j) a_{i+j}^{(m,k)} \left[m^i + \cdots\right].
\end{aligned}$$

The coefficients of x^0 have to be equal, from which follows

$$a_j^{(m,m-k-1)} = \frac{(-1)^j}{j!} \sum_{i=0}^{m-j-1} (i+1)(i+2)\cdots(i+j) a_{i+j}^{(m,k)} m^i, \quad 0 \le j \le m-2.$$

It is easy to verify that the last relation can be extended to the case $j = m-1$, and therefore, the leading coefficient can be calculated by

$$a_{m-1}^{(m,m-k-1)} = (-1)^{m-1} a_{m-1}^{(m,k)}.$$

According to the previous consideration we are able to formulate the following result.

Theorem 2.1. *Let $m \in \mathbb{N}$, $0 \le k \le m-1$, $x \in [k, k+1]$ and let*

$$\varphi_m(x) = \sum_{i=0}^{m-1} a_i^{(m,k)} x^i.$$

Furthermore, let the coefficients of the cardinal B-spline of order $m-1$ be known. The coefficients of the cardinal B-spline of order m satisfy the following relations:

$$a_{m-1}^{(m,0)} = \frac{1}{m-1} a_{m-2}^{(m-1,0)}; \tag{2.4}$$

$$a_i^{(m,0)} = 0, \quad m-2 \ge i \ge 0. \tag{2.5}$$

1° *If m is odd and $1 \le k \le [m/2]$ ($[a]$ denotes the largest integer not greater than the real number a), i.e., if m is even and $1 \le k \le m/2 - 1$, then*

$$a_{m-1}^{(m,k)} = \frac{1}{(m-1)(m-k)} \left[m a_{m-2}^{(m-1,k)} - k(m-1) a_{m-1}^{(m,k-1)} - a_{m-2}^{(m,k-1)}\right], \tag{2.6}$$

$$a_i^{(m,k)} = \frac{m}{i+1-m} \left[(i+1) a_{i+1}^{(m,k)} - a_i^{(m-1,k)}\right], \quad m-2 \ge i \ge 0; \tag{2.7}$$

2° *If m is odd and $[m/2]+1 \le k \le m-1$, i.e., if m is even and $m/2 \le k \le m-1$, then*

$$a_{m-1}^{(m,m-k-1)} = (-1)^{m-1} a_{m-1}^{(m,k)}, \tag{2.8}$$

$$a_j^{(m,m-k-1)} = \frac{(-1)^j}{j!} \sum_{i=0}^{m-j-1} (i+1)(i+2)\cdots(i+j) a_{i+j}^{(m,k)} m^i, \quad 0 \le j \le m-2. \tag{2.9}$$

It is easy to check that the relation (2.6), in the case $k = 0$, under assumptions $a_{m-1}^{(m,-1)} = 0$ and $a_{m-2}^{(m,-1)} = 0$, becomes (2.4). Furthermore, under the same assumptions, relation (2.7) becomes (2.5).

Regarding the previous result we can formulate the following algorithm for calculating the coefficients of the cardinal B-spline.

The input data in the algorithm is the order of the cardinal B-spline, positive integer M. The output data are coefficients of the cardinal B-splines of orders $1, 2, \ldots, M$. The obtained coefficients are elements of the three dimensional array a, where we used tag $a(i, m, k) = a_i^{(m,k)}$. The algorithm is realized in such a way that the coefficients of the polynomials defined on the symmetric intervals are calculated in the same loop. If the number of intervals is odd, the coefficients of the polynomial defined on the middle interval are calculated in the last step. The described procedure repeats M times.

Algorithm

Set $a(0, 1, 0) = 1$.

for $m = 2$ **to** M **do**
  Set $a(m-1, m, -1) = 0$, $a(m-2, m, -1) = 0$ and $g = \left[\frac{m}{2}\right] - 1$.
 for $k = 0$ **to** g **do**
   to calculate coefficient $a(m-1, m, k)$ use equality (2.6).
   to calculate coefficient $a(m-1, m, m-k-1)$ use equality (2.8).
  for $i = m-2$ **to** 0 **do**
    to calculate coefficient $a(i, m, k)$ use equality (2.7).
    to calculate coefficient $a(i, m, m-k-1)$, set $j = i$, rename index of summation and use equality (2.9).
if m **is odd then**
  to calculate coefficient $a(m-1, m, g+1)$ set $k = g+1$ and use equality (2.6).
  for $i = m-2$ **to** 0 **do**
    to calculate coefficient $a(i, m, g+1)$ set $k = g+1$ and use equality (2.7).

3. Numerical results

By using the described algorithm, we calculate the coefficients of the cardinal B-splines of orders $2, 3, \ldots, 7$. The corresponding coefficients are given in the matrix form since that cardinal B-spline is completely determined in that way. Namely, the matrix product

$$\begin{bmatrix} a_{m-1}^{(m,0)} & a_{m-2}^{(m,0)} & \cdots & a_0^{(m,0)} \\ a_{m-1}^{(m,1)} & a_{m-2}^{(m,1)} & \cdots & a_0^{(m,1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1}^{(m,m-1)} & a_{m-2}^{(m,m-1)} & \cdots & a_0^{(m,m-1)} \end{bmatrix} \begin{bmatrix} x^{m-1} \\ x^{m-2} \\ \vdots \\ 1 \end{bmatrix}$$

is equal to the column whose components are the corresponding polynomials. The cardinal B-spline of order m is equal to the first polynomial at the interval $[0, 1]$, to the second polynomial at the interval $[1, 2]$, etc. The cardinal B-spline of order m is equal to the last polynomial at the interval $[m-1, m]$.

Now we list results for $m = 2, 3, \ldots, 7$.

Case $m = 2$. For the cardinal B-spline of second order, the corresponding matrix is

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

Case $m = 3$. For the cardinal B-spline of third order, the corresponding matrix is

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 3 & -\frac{3}{2} \\ \frac{1}{2} & -3 & \frac{9}{2} \end{bmatrix}.$$

Case $m = 4$. For the cardinal B-spline of order 4, the corresponding matrix is

$$\begin{bmatrix} \frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{2} & 2 & -2 & \frac{2}{3} \\ \frac{1}{2} & -4 & 10 & -\frac{22}{3} \\ -\frac{1}{6} & 2 & -8 & \frac{32}{3} \end{bmatrix}.$$

Case $m = 5$. For the cardinal B-spline of order 5, the corresponding matrix is

$$\begin{bmatrix} \frac{1}{24} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{5}{6} & -\frac{5}{4} & \frac{5}{6} & -\frac{5}{24} \\ \frac{1}{4} & -\frac{5}{2} & \frac{35}{4} & -\frac{25}{2} & \frac{155}{24} \\ -\frac{1}{6} & \frac{5}{2} & -\frac{55}{4} & \frac{65}{2} & -\frac{655}{24} \\ \frac{1}{24} & -\frac{5}{6} & \frac{25}{4} & -\frac{125}{6} & \frac{625}{24} \end{bmatrix}.$$

Case $m = 6$. For the cardinal B-spline of order 6, the corresponding matrix is

$$\begin{bmatrix} \frac{1}{120} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{24} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{20} \\ \frac{1}{12} & -1 & \frac{9}{2} & -\frac{19}{2} & \frac{39}{4} & -\frac{79}{20} \\ -\frac{1}{12} & \frac{3}{2} & -\frac{21}{2} & \frac{71}{2} & -\frac{231}{4} & \frac{731}{20} \\ \frac{1}{24} & -1 & \frac{19}{2} & -\frac{89}{2} & \frac{409}{4} & -\frac{1829}{20} \\ -\frac{1}{120} & \frac{1}{4} & -3 & 18 & -54 & \frac{324}{5} \end{bmatrix}.$$

Case $m = 7$. For the cardinal B-spline of order 7, the corresponding matrix is

$$\begin{bmatrix} \frac{1}{720} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{120} & \frac{7}{120} & -\frac{7}{48} & \frac{7}{36} & -\frac{7}{48} & \frac{7}{120} & -\frac{7}{720} \\ \frac{1}{48} & -\frac{7}{24} & \frac{77}{48} & -\frac{161}{36} & \frac{329}{48} & -\frac{133}{24} & \frac{1337}{720} \\ -\frac{1}{36} & \frac{7}{12} & -\frac{119}{24} & \frac{196}{9} & -\frac{1253}{24} & \frac{196}{3} & -\frac{12089}{360} \\ \frac{1}{48} & \frac{7}{12} & \frac{161}{24} & \frac{364}{9} & \frac{3227}{24} & -\frac{700}{3} & \frac{59591}{360} \\ -\frac{1}{120} & \frac{7}{24} & -\frac{203}{48} & \frac{1169}{36} & -\frac{6671}{48} & \frac{7525}{24} & -\frac{208943}{720} \\ \frac{1}{720} & -\frac{7}{120} & \frac{49}{48} & -\frac{343}{36} & \frac{2401}{48} & -\frac{16807}{120} & \frac{117649}{720} \end{bmatrix}.$$

References

[1] C. Chui, An Introduction to Wavelets, Academic Press, Boston, 1992.
[2] C. Chui, Wavelets: A Mathematical Tool for Signal Analysis, SIAM, Philadelphia, 1997.